SOLVING NONLINEAR PROGRAMMING PROBLEMS BY LINEAR APPROXIMATIONS

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Abstract

By dividing the domain of a function into smaller intervals, one can find linear approximations of the function. It was used these to linearize a particular type of nonlinear programming problems, in order to use a modified simplex algorithm to give an approximate solution of such problems.

Keywords: nonlinear programming, linear approximation

Introduction

Suppose that \( f : [a, b] \to \mathbb{IR} \) (1) is a continuous function. For any \( x \in [a, b] \), there is a unique \( \lambda \in [0,1] \) such that

\[
x = \lambda \cdot a + (1 - \lambda) \cdot b
\]  

Denoting \( \mu = 1 - \lambda \), one can then approximate the function \( f \) with

\[
f(x) \approx \lambda \cdot f(a) + \mu \cdot f(b)
\]  

where \( \lambda + \mu = 1 \), which is linear with respect to \( \lambda \) and \( \mu \).

This approximation is efficient only for small intervals (Bellman, 1967; Marusciac, 1973). Generally, one can divide the interval \([a, b]\) into a number of smaller intervals by choosing a division

\[
a = x_0 < x_1 < \ldots < x_{k-1} < x_k < \ldots < x_{n-1} < x_n = b
\]  

Every point \( x \in [a, b] \) can then be written as

\[
x = \lambda_0 \cdot x_0 + \lambda_1 \cdot x_1 + \ldots + \lambda_n \cdot x_n
\]  

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where \( \lambda_i \in [0,1] \), for each \( i = \overline{1,n} \), such that
\[
\lambda_0 + \lambda_1 + \ldots + \lambda_n = 1 \tag{6}
\]
and no more than two consecutive of the \( \lambda_i \)'s are nonzero (to be precise, we have \( \lambda_{k-1}, \lambda_k \neq 0 \Leftrightarrow x \in [x_{k-1}, x_k] \)). The corresponding approximation of \( f \) is
\[
f(x) \approx \lambda_0 \cdot f(x_0) + \lambda_1 \cdot f(x_1) + \ldots + \lambda_n \cdot f(x_n) \tag{7}
\]
with
\[
\lambda_0 + \lambda_1 + \ldots + \lambda_n = 1
\]
and no more than two consecutive \( \lambda_i \)'s nonzero (Malita, 1971).

**Results and Discussion**

Suppose now that

\[
\begin{cases}
\text{opt}[f(x_1, x_2, \ldots, x_m)] \\
g_1(x_1, x_2, \ldots, x_m) \leq (\geq) 0 \\
\vdots \\
g_r(x_1, x_2, \ldots, x_m) \leq (\geq) 0 \\
x_k \in [a_k, b_k], k = \overline{1,m}
\end{cases} \tag{8}
\]

is a programming problem with nonlinear objective function \( f \) and/or nonlinear restriction functions \( g_1, \ldots, g_r \), such that
\[
f(x_1, x_2, \ldots, x_m) = f_1(x_1) + f_2(x_2) + \ldots + f_m(x_m) \tag{9}
\]
and
\[
g_i(x_1, x_2, \ldots, x_m) = g_{i1}(x_1) + g_{i2}(x_2) + \ldots + g_{im}(x_m) \tag{10}
\]
for any \( i = \overline{1,r} \) (Postelniciu, 1977).

Linearizing each of the functions \( f_k, g_{ik} \) as described in the first section, could be obtain a linear programming problem:
with the supplementary restriction that only two consecutive elements of each set of $\lambda_{ki}$’s, $k = 1, m$, may be nonzero.

One can solve this linear programming problem using a modified simplex algorithm, such that at each iteration step the supplementary restriction holds.

Of course, the optimal solution of the linear programming problem is not necessarily optimal for the initial nonlinear problem, but it represents, still, an approximation of an optimal solution. Other method may then allow to further optimizing this approximate solution.

In the sequel, we shall see, by hand of an example, how this linearization method works (Otiman, 2002; Vaduva, 1974).

Consider the following nonlinear programming problem

$$\begin{align*}
\max & \{ f(x_1, x_2) = 3x_1^3 + 2x_2^3 \} \\
x_1^2 + x_2^2 & \leq 16 \\
x_1 - x_2 & \leq 3 \\
x_1 & \geq 0, x_2 \geq 0
\end{align*}$$

Under the given conditions, it is obvious that $x_1, x_2 \in [0, 4]$. The problem could be linearised as described in the previous section, using the divisions of $[0, 4]$: $x_{10} = 0, x_{11} = 1, x_{12} = 2, x_{13} = 3, x_{14} = 4$. (13)
and
\[ x_{20} = 0, x_{21} = 1, x_{22} = 2, x_{23} = 3, x_{24} = 4. \] (14)

The objective function can be written as
\[ f(x_1, x_2) = f_1(x_1) + f_2(x_2), \] (15)
with: \( f_1(x_1) = 3x_1^3, \quad f_2(x_2) = 2x_2^3. \)

Also
\[ g_1(x_1, x_2) = x_1^2 + x_2^2 - 16 = g_{11}(x_1) + g_{12}(x_2) - 16, \] (16)
where: \( g_{11}(x_1) = x_1^2, g_{12}(x_2) = x_2^2. \)

One obtains the linear programming problem:
\[
\begin{align*}
\text{max} [z &= 3\lambda_{11} + 24\lambda_{12} + 81\lambda_{13} + \\
&+ 192\lambda_{14} + 2\lambda_{21} + 16\lambda_{22} + 54\lambda_{23} + 128\lambda_{24}] \\
\lambda_{11} + 4\lambda_{12} + 9\lambda_{13} + 16\lambda_{14} + \\
&+ \lambda_{21} + 4\lambda_{22} + 9\lambda_{23} + 16\lambda_{24} \leq 16 \\
\lambda_{11} + 2\lambda_{12} + 3\lambda_{13} + 4\lambda_{14} - \\
&- \lambda_{21} - 2\lambda_{22} - 3\lambda_{23} - 4\lambda_{24} \leq 3 \\
\lambda_{10} + \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} = 1, \\
\lambda_{20} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} = 1, \\
\lambda_{1j} \geq 0, \lambda_{2j} \geq 0, j = 0, 4, \\
\end{align*}
\] (17)
with the supplementary restriction, that no more than two consecutive of the \( \lambda_{1j} \)’s and \( \lambda_{2j} \)’s may be nonzero.

Using the modified simplex method, such that the supplementary condition is always fulfilled, one obtains the following solutions:
\[ \lambda_{10} = 0, \lambda_{11} = 0, \lambda_{12} = 0, \lambda_{13} = \frac{1}{8}, \lambda_{14} = \frac{7}{8}, \]
\[ \lambda_{10} = \frac{1}{8}, \lambda_{11} = \frac{7}{8}, \lambda_{12} = 0, \lambda_{13} = 0, \lambda_{14} = 0, \]
for which
\[ z_{\text{max}} = \frac{81}{8} \cdot \frac{1}{8} + 192 \cdot \frac{7}{8} + 0 \cdot \frac{1}{8} + 2 \cdot \frac{7}{8} = \frac{1439}{8}. \]

From this, we obtained an approximate solution of the initial nonlinear programming problem:

\[ x_1 = 0 \cdot \lambda_{10} + 1 \cdot \lambda_{11} + 2 \cdot \lambda_{12} + 3 \cdot \lambda_{13} + 4 \cdot \lambda_{14} = \frac{31}{8}, \]

\[ x_2 = 0 \cdot \lambda_{20} + 1 \cdot \lambda_{21} + 2 \cdot \lambda_{22} + 3 \cdot \lambda_{23} + 4 \cdot \lambda_{24} = \frac{7}{8}, \]

with the corresponding value of the objective function

\[ f\left(\frac{31}{8}, \frac{7}{8}\right) = 3 \cdot \left(\frac{31}{8}\right)^3 + 2 \cdot \left(\frac{7}{8}\right)^3 = \frac{90059}{512}. \]

**Conclusions**

By linearizing a nonlinear programming problem, one has the possibility of applying well-known algorithms, which can solve the associated linear programming problem. The optimal solution of the linear programming problem leads to a solution of the initial problem, which is not necessarily an optimal solution, but an approximation of the optimal one. The quality of the approximation depends on the approximation quality of the nonlinear functions:

\[ f_1, \ldots, f_m, g_{11}, \ldots, g_{1m}, \ldots, g_{r1}, \ldots, g_{rm}. \]

This may increase if one refines the divisions of the intervals \([a_k, b_k], k = 1, m\).

The problem with such a refinement is that one obtains a large number of variables \(\lambda_{ki}\). On the other hand, this is not much of a problem for good mathematical software, hence one can rely on computers to do the job and give a good approximate solution of the initial nonlinear programming problem.
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References


