APPROXIMATING FUNCTIONS WITH CHEBYSHEV POLYNOMIALS

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Abstract

In this paper, are reminded the most important, from the point of view of approximation theory, properties of Chebyshev polynomials and give a short Maple procedure to compute the coefficients of the Chebyshev-Fourier Series expansion of a continuous function on the interval [-1,1].

Key-words: Chebyshev polynomials, Chebyshev-Fourier Series

Introduction

One common way of approximating functions is to use Taylor series expansions. This relies on the computation of the Taylor polynomials of the function up to a certain order, and approximating the given function through these Taylor polynomials (Iaglom, 1983; Mocica, 1988). While this is a relatively simple procedure in case of smooth functions, it cannot work for nondifferentiable continuous functions. Also the convergence of these approximations is not uniformly distributed on a given interval, towards the ends of the intervals the approximation errors being higher (Panaitopol, 1980).

In order to avoid these problems, one can use different families of orthogonal polynomials – like Chebyshev’s, Laguerre’s, Legendre’s or Hermite’s (Dancea, 1973; Rudner, 1982). In the sequel, could be consider Chebyshev polynomials of the first kind (there is also a family of Chebyshev polynomials of the second kind).


Results and Discussion

The Chebyshev polynomials of the first kind are known also as the optimal approximation polynomials on the interval \([-1, 1]\). They are defined as

\[
T_n(x) = \cos(n \cdot \arccos(x)), \quad n \in \mathbb{N}, x \in [-1,1].
\]  

(1)

With \(x = \cos(\alpha)\), expanding \(\cos(n \cdot \alpha)\), one can find that

\[
T_n(x) = \sum_{k=0}^{\left[ \frac{n}{2} \right]} C_n^{2k} \cdot x^{n-2k} \cdot \left( x^2 - 1 \right)^k,
\]  

or

\[
T_n(x) = \sum_{k=0}^{\left[ \frac{n}{2} \right]} (-1)^k \cdot x^{n-2k} \cdot \sum_{l=k}^{n} C_n^{2k} \cdot C_l^k.
\]  

(2)

(3)

In order to compute the Chebyshev polynomials of the first kind one can use also Rodriguez’s formula:

\[
T_n(x) = (-1)^n \frac{\sqrt{1-x^2}}{(2n-1)!!} \cdot \frac{d^n}{dx^n} (1-x^2)^{n-\frac{1}{2}},
\]  

(4)

or the generating function

\[
\sum_{n=0}^{\infty} T_n(x) \cdot t^n.
\]  

(5)

Another simple way of constructing Chebyshev’s polynomials relies on the recurrence relation

\[
T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad (\forall) n \geq 2,
\]  

(6)

starting with \(T_0(x) = 1\), and \(T_1(x) = x, (\forall) x \in [-1,1]\).

Most useful for the approximation theory is the fact that Chebyshev’s polynomials of the first kind form a complete set of orthogonal polynomials with respect to the weight function

\[\rho(x) = \sqrt{1-x^2}\]:
The Chebyshev polynomials expansion, or Chebyshev-Fourier series expansion, of a function \( f \) on the interval \([-1, 1]\) is then given by

\[
f(x) = \sum_{n=0}^{\infty} \alpha_n \cdot T_n(x),
\]

where

\[
\alpha_0 = \frac{1}{\pi} \cdot \frac{1}{\sqrt{1 - x^2}} \int_{-1}^{1} f(x) \, dx,
\]

and

\[
\alpha_n = \frac{2}{\pi} \cdot \frac{1}{\sqrt{1 - x^2}} \int_{-1}^{1} f(x) \cdot T_n(x) \, dx, \quad (\forall) n > 0.
\]

The following Maple procedure allows the computation of the Chebyshev-Fourier Series coefficients up to a certain specified order \( n \), and computes the approximation of a function \( f \) using Chebyshev’s polynomials up to order \( n \).

> Cheb_approx:=proc(f,n) local F,i,a;
F:=0;
for i from 0 to n do
if i=0
then a:=1/Pi*int(1/sqrt(1-x^2)*f(x),x=-1..1)
else a:=2/Pi*int(1/sqrt(1-x^2)*f(x)*orthopoly[T](i,x),x=-1..1)
fi;
F:=F+a*orthopoly[T](i,x)
end;
F
end;

Let us apply this procedure in order to compute the 5\(^{th}\) Chebyshev approximation of the polynomial \( X^7 + 2 \cdot X^4 + 3 \cdot X - 2 \)
**Approximating Functions with Chebyshev Polynomials**

> g:=unapply(Cheb_approx(x->x^7+2*x^4+3*x-2,5),x);

\[ g := x \mapsto -2 + \frac{199}{64} x - \frac{7}{8} x^3 + 2 x^4 + \frac{7}{4} x^5 \]

If is plotted (figure 1) the difference between the approximation function and the given polynomial:

\[ \text{plot}(g-(x->x^7+2*x^4+3*x-2),-1..1); \]

could be seen that this difference is very small (in our case it’s absolute value is less than 0.016)

![Fig. 1. Difference between the approximation function and given polynomial](image-url)

**Conclusions**

The approximation of a function using Chebyshev series expansion is of real interest, known the fact that Chebyshev’s polynomials are the polynomials of best approximation on the interval \([-1, 1]\).

**References**